Fourier Analysis Through Examples Using Wolfram Mathematica

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Abstract - The representation of a function in the form of a series is fairly common practice in mathematics. Fourier series is an expansion of a periodic function $f(x)$ in which base set is the set of sine and cosine functions. In attempt to define the Fourier series of a nonperiodic functions is obtained the Fourier transform, as a continuous representation. In this paper we provide the Fourier series on several functions, define on $[-\pi,\pi]$, $[-l,l]$, or $[a,b]$. Also, through examples we discuss whether the Fourier transform of some function exist or not, and we consider some properties, such as linearity; Fourier transform of the operator for modulation, translation and time-frequency shift. Then using the mathematical package Wolfram Mathematica, we visually present the results that we obtain for Fourier series and Fourier transform, which in fact is real and complex function, respectively.

I. INTRODUCTION

Fourier analysis deals with analysis of a functions in terms of and in relation with Fourier series expansion and Fourier transform. The idea of expanding a function in the form of series, in which the base is the set of sine and cosine functions, was given by French physicist Joseph Fourier in 1807 as a result of necessity to solve practical problems in physics, such as heat-flow problems, wave propagation and diffusion. The Fourier series expansion first was given for a periodic function, then in an attempt to define it of a nonperiodic functions is obtained the Fourier transform, as a continuous representation. Even more, the Fourier transform can be considerate as a ready-made tool, which has application in several areas, such as mathematics for solving differential equation; signal and system analysis. More about Fourier analysis one can find in [1,3,4, 6].

In this paper we make a short review of the fundamental results for Fourier series and Fourier transform. Then we provide the Fourier series of continuous function, function with one finite discontinuity and continuous function extended to an odd function (see example 1, 2, 3). The Fourier transform of an odd and even function, and translation and modulation of some function is given in example 4, 5 and 6, respectively. In all examples we visually present the obtained results, using the mathematical package Wolfram Mathematica.

II. PRELIMINARIES

A. Fourier series expansion

Let $f(x)$ is 2π-periodic function. If the trigonometric series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad (1)$$

converges uniformly on $(-\pi,\pi)$, and its sum is the function $f(x)$, i.e.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),\ x\in(-\pi,\pi),$$

then for coefficients a_0, a_n and b_n hold

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx, \qquad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx.$$

The coefficients a_0, a_n and b_n are called Fourier coefficients, and the series (1) is Fourier series expansion. The corresponding exponential Fourier series expansion is

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx},$$

where $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx,\ n\in\mathbb{Z}$.

If $f(x)$ is an even function on interval $(-\pi,\pi)$, then its Fourier series expansion is $\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$, $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$, n=0,1,..; and if it is an odd function, its Fourier serie expansion is $\sum_{n=1}^{\infty}b_n\sin nx$, $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nxdx$, $n=1,2,...$.

Dirichlet's Theorem: Let $f(x)$ is defined on $[-\pi,\pi]$. If the function $f(x)$ satisfied the Dirichlet's conditions on $[-\pi,\pi]$, i.e. it has a finite number of isolated maxima and minima, and a finite number of points of finite discontinuity, then the Fourier series expansion of $f(x)$ uniformly converges to $S(x)$ at all points on $[-\pi,\pi]$, for which hold:

1. $S(x)=f(x)$, where $f(x)$ is continuous;
2. At the points of discontinuity hold
$$S(x)=\frac{f(x-0)+f(x+0)}{2};$$
3. At the end points of $(-\pi,\pi)$ hold
$$S(-\pi)=S(\pi)=\frac{f(-\pi+0)+f(\pi-0)}{2}.$$

The same theorem hold if the function $f(x)$ is defined and satisfied the Dirichlet's condition on $(-\pi,\pi)$, $(-\pi,\pi]$, $[-\pi,\pi)$, $[a,b]$, (a,b), $(a,b]$ or $[a,b)$, $a,b\in\mathbb{R}$.

If the function $f(x)$ is defined on $[-l,l]$, and satisfied the Dirichlet's conditions then its Fourier series has the form (1), such that the argument in sine and cosine functions is $\frac{n\pi x}{l}$ instead of nx, and the integration is from $-l$ to l. If a function is defined on $[a,b]$, then $l=\frac{b-a}{2}$.

B. Fourier transform

Let $f(x)$ is periodic function with period l, $l\in\mathbb{R}^+$, which is defined and satisfied Dirichlet's condition on $[-l/2, l/2]$. Its exponential Fourier series expansion is

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi inx}{l}}, \qquad (2)$$

where

$$c_n=\frac{1}{l}\int_{-l/2}^{l/2} f(x)e^{\frac{-2\pi inx}{l}}dx,\ n\in\mathbb{Z}.$$

If we consider the coefficients c_n as a function F from variable $\frac{n}{l}$, i.e.

$$F\left(\frac{n}{l}\right)=\int_{-l/2}^{l/2} f(x)e^{\frac{-2\pi inx}{l}}dx, \qquad (3)$$

then for the Fourier series expansion (2) we obtain

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{1}{l}F\left(\frac{n}{l}\right)e^{\frac{2\pi inx}{l}}. \qquad (4)$$

Now, if l tends to infinity in (3), the range of $f(x)$ will be from $-\infty$ to ∞, and let we redefine n to be the "frequency," which we denote with ω, i.e.

$$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-2\pi i\omega x}dx. \qquad (5)$$

The function $F:\mathbb{R}\to\mathbb{C}$ is called the Fourier transform of the function $f(x)$, and we will use the notation $\hat{f}$ or $\mathcal{F}$ when we consider it as an operator. The Fourier transform (5) exist whenever $\int_{-\infty}^{\infty}|f(x)|dx<\infty$. When l tends to infinity, the continuous form of (4) is $f(x)=\int_{-\infty}^{\infty}F(\omega)e^{2\pi i\omega x}d\omega$, such called the inverse Fourier transform.

If $f(x)$ is an even function on $(-\infty,\infty)$, it's Fourier transform is $\hat{f}(\omega)=2\int_{0}^{\infty}f(x)\cos 2\pi x\omega$, and if it is an odd, then $\hat{f}(\omega)=-2i\int_{0}^{\infty}f(x)\sin 2\pi x\omega$.

The linearity of the Fourier transform is an important property. For $b,a\in\mathbb{R}$, with $T_b f(x)=f(x-b)$ and $M_a f(x)=e^{2\pi iax}f(x)$ we denote the operator for translation and modulation, respectively, and with $T_b\,M_a$ and $M_a\,T_b$ the time frequency operators, such that $T_bM_a f(x)=e^{-2\pi iba}M_aT_b f(x)$. Even more, one can prove that $\mathcal{F}(T_b f)=M_{-b}\hat{f}$, $\mathcal{F}(M_a f)=T_a\hat{f}$, $\mathcal{F}(T_bM_a f)=e^{-2\pi iba}T_aM_{-b}\hat{f}$, (see e.g. [1]).

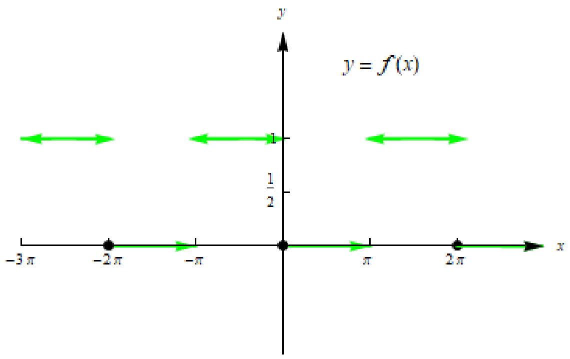
III. MAIN RESULTS

In this section through examples we visually present the Fourier series expansion of continuous function and function with points of discontinuity defined on different interval (see example 1, 2, 3). Also, we provide the Fourier transform of an odd, even function and translation and modulation of some function, (see example 4, 5, 6). In example 7 we consider functions for which Fourier transform does not exist.

Example 1. Let $f(x)=\begin{cases}1, & -\pi<x<0\\ 0, & 0\le x<\pi\end{cases}$ is 2π-periodic function. By Dirichlet's theorem the Fourier series expansion is

$$S(x)=\frac{1}{2}-\frac{2}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right),$$

equal to $f(x)$ for all $x\in\mathbb{R}\setminus\{k\pi, k\in\mathbb{Z}\}$ (see Fig.1).



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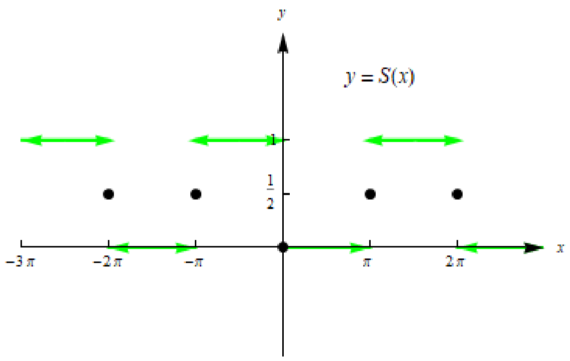

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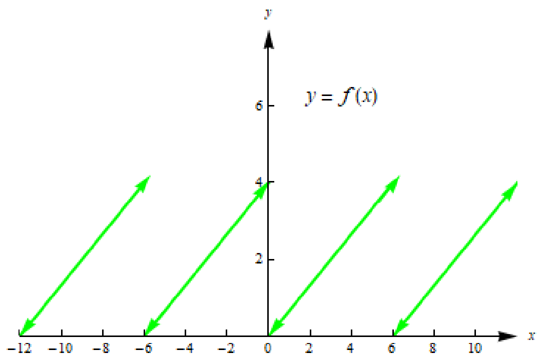
Figure 1. a) Function $f(x)=\begin{cases}1, & -\pi<x<0\\ 0, & 0\le x<\pi\end{cases}$; b) Its Fourier series expansion.

Example 2. Let $f(x)=\frac{2x}{3}$, with period 6 on $(0,6)$. By Dirichlet's theorem the exponential Fourier series expansion is

$$S(x)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{-2}{in\pi}e^{\frac{in\pi x}{3}},$$

equal to $f(x)$ for all $x\in\mathbb{R}\setminus\{6k, k\in\mathbb{Z}\}$ (see Fig.2).



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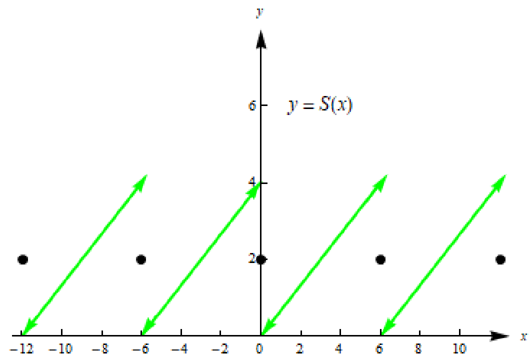

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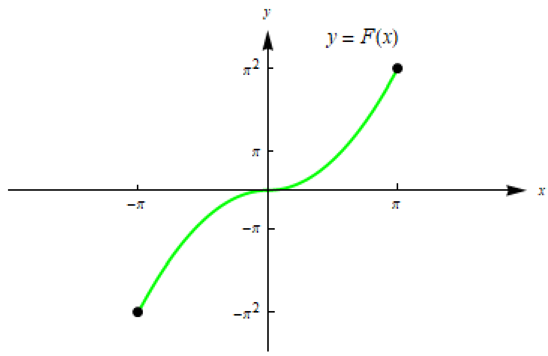
Figure 2. a) Function $f(x)=\frac{2x}{3}$; b) Its Fourier series expansion.

Example 3. Let $f(x)=x^2$ for $x\in[0,\pi]$. We define the odd function $F(x)=\begin{cases}x^2, & x\in[0,\pi]\\ -x^2, & x\in[-\pi,0]\end{cases}$.

By Dirichlet's theorem the Fourier series expansion of the function $F(x)$ is

$$S(x)=2\pi\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}, \quad (6)$$

$x\in(-\pi,\pi)$. Because $F(x)$ is extension of $f(x)$, the Fourier series expansion of the function $f(x)$ is the same function in (6), but for $x\in(0,\pi)$ (see Fig.3).



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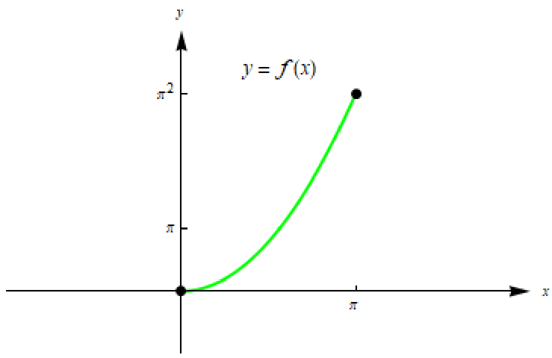

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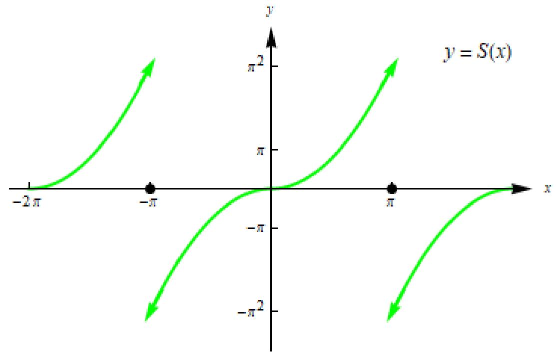


c)

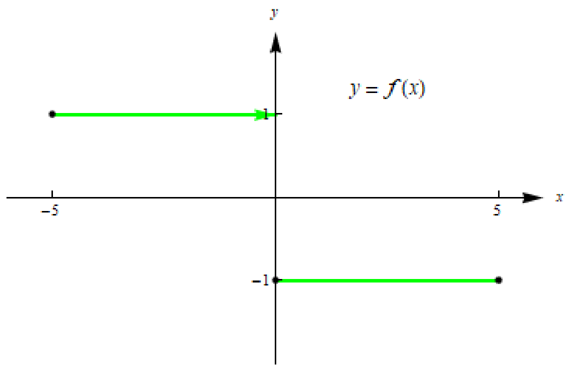
Figure 3. Function a) $f(x)=x^2$; b) $F(x)=\begin{cases} x^2, & x\in[0,\pi] \\ -x^2, & x\in[-\pi,0] \end{cases}$;

c) Fourier series expansion of $f(x)$ and $F(x)$.

Example 4. The Fourier transform of the function $f(x)=\begin{cases} A, & x\in[-T,0) \\ -A, & x\in[0,T] \end{cases}$, $A,\ T>0$ is

$$\hat{f}(\omega)=2Ai\int_0^T \sin(2\pi\omega x)dx \;=\frac{Ai}{\pi\omega}\left(1-\cos(2\pi\omega T)\right).$$

On Fig.4 is given the graphic of the function and the imaginary part of its Fourier transform for $A=1$ and $T=5$.



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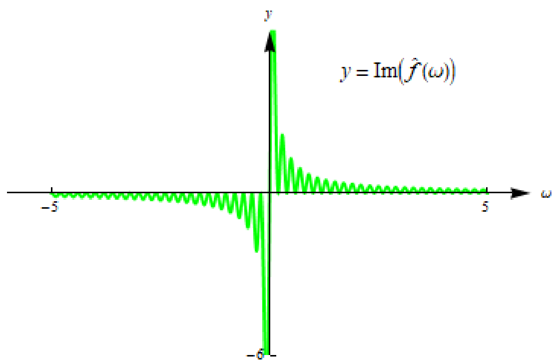

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Figure 4. a) Function $f(x)=\begin{cases} 1, & x\in[-5,0) \\ -1, & x\in[0,5] \end{cases}$;

b) Imaginary part of its Fourier transform.

Example 5. For the Fourier transform of the function

$$f(x)=\begin{cases} \cos 3x, & -\pi\le x\le\pi \\ 0, & \text{otherwise} \end{cases}$$

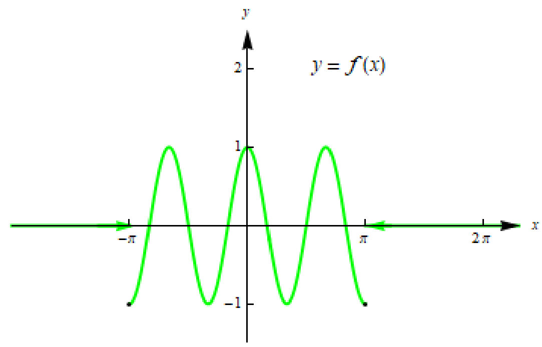
we obtain

$$\begin{aligned}\hat{f}(\omega) &= \int_0^{\pi}(\cos(3+2\pi\omega)x+\cos(3-2\pi\omega)x)dx \\ &= \frac{\sin(3+2\pi\omega)\pi}{3+2\pi\omega}+\frac{\sin(3-2\pi\omega)\pi}{3-2\pi\omega} \\ &= \frac{6\sin 3\pi\cos 2\pi^2\omega-4\pi\omega\cos 3\pi\sin 2\pi^2\omega}{9-4\pi^2\omega^2} \\ &= \frac{4\pi\omega\sin 2\omega\pi^2}{9-4\pi^2\omega^2}.\end{aligned}$$

On Fig.5 one can see the graphic of the function and its Fourier transform.

Example 6. Let $f(x)=\begin{cases} 1, & x\in[1,2] \\ 0, & \text{otherwise} \end{cases}$. The translation and modulation of the function $f(x)$ for $b\in\mathbb{R}$ and $a\in\mathbb{R}$, is defined as $T_b f(x)=\begin{cases} 1, & x\in[1+b,2+b] \\ 0, & \text{otherwise} \end{cases}$ and

$M_a f(x)=\begin{cases} e^{2\pi i x a}, & x\in[1,2] \\ 0, & \text{otherwise} \end{cases}$, respectively.



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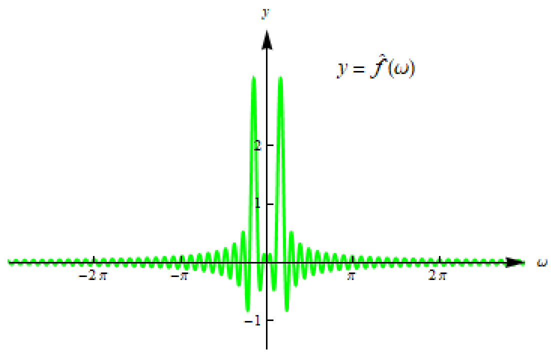
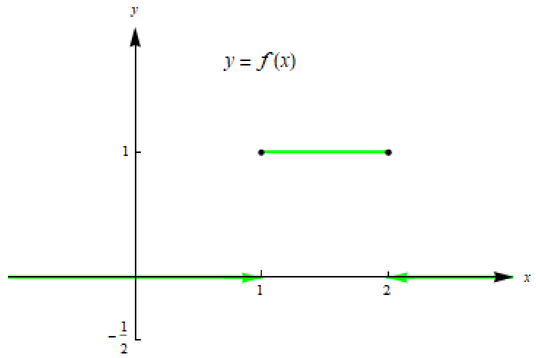


b)

Figure 5. a) Function $f(x)=\begin{cases}\cos 3x, & -\pi \le x \le \pi \\ 0, & \text{otherwise}\end{cases}$; b) Its Fourier transform.

On Fig.6 and 7 one can find the translation for $b=3$ and $b=-3$, and the real and imaginary part of the modulation for $a=2$ and $a=4$ of the function $f(x)=\begin{cases}1, & x\in[1,2] \\ 0, & \text{otherwise}\end{cases}$, respectively.



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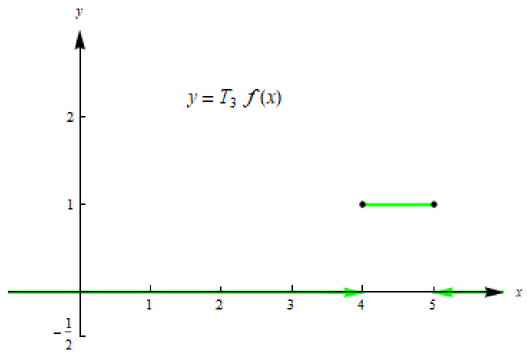

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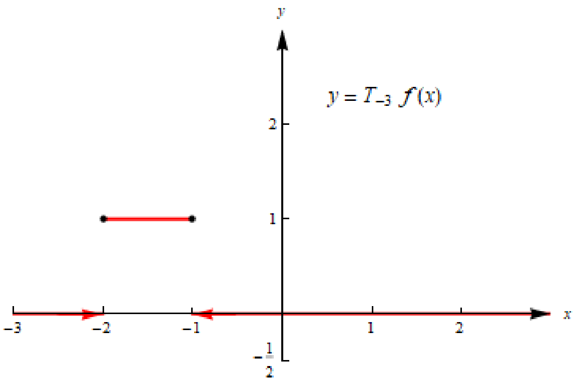
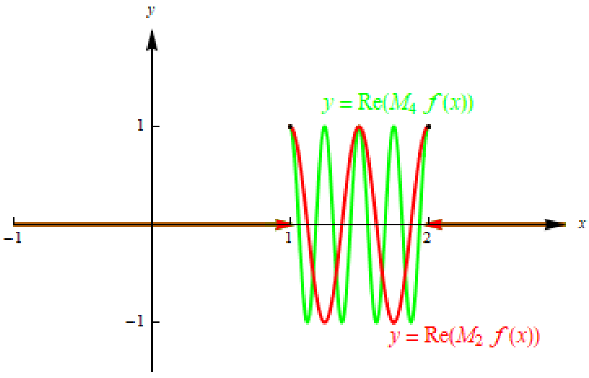


c)

Figure 6. a) Function $f(x)=\begin{cases}1, & x\in[1,2] \\ 0, & \text{otherwise}\end{cases}$; Translation of $f(x)$ b) for $b=3$; c) for $b=-3$.

It is clear that the Fourier transform of functions $f(x)$, $T_b f(x)$, $M_a f(x)$, $b,\ a\in\mathbb{R}$ exist.



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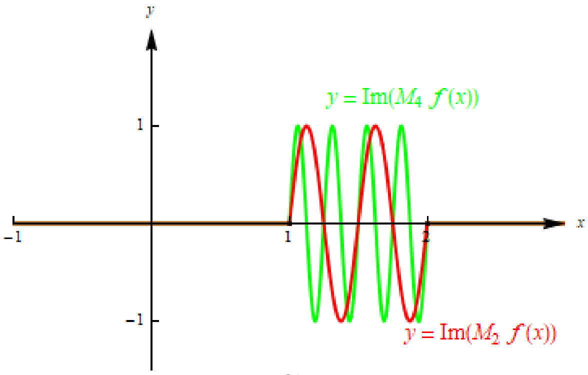


b)

Figure 7. a) Real; b) Imaginary part of modulation of the function $f(x)=\begin{cases}1, & x\in[1,2] \\ 0, & \text{otherwise}\end{cases}$ for $a=2$ and $a=4$.

For the Fourier transform (see Fig. 8) of the function $f(x)$ we obtain

$$\hat{f}(\omega)=\int_1^2 e^{-2\pi i x\omega}dx=-\frac{e^{-4\pi i\omega}}{2\pi i\omega}+\frac{e^{-2\pi i\omega}}{2\pi i\omega}$$

$$=i\,\frac{e^{-4\pi i\omega}-e^{-2\pi i\omega}}{2\pi\omega}$$

$$=\frac{\cos(3\pi\omega)\sin(\pi\omega)}{\pi\omega}-i\,\frac{\sin(3\pi\omega)\sin(\pi\omega)}{\pi\omega}.$$

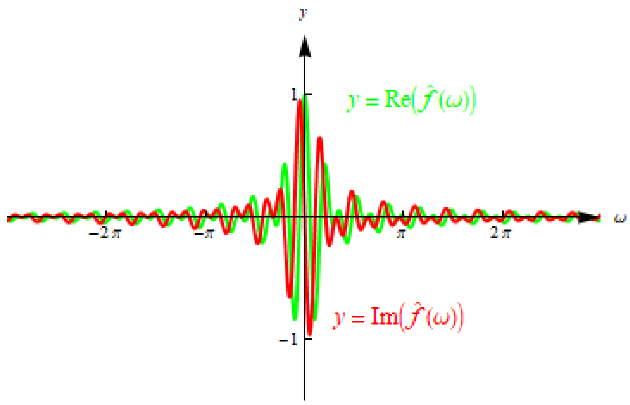
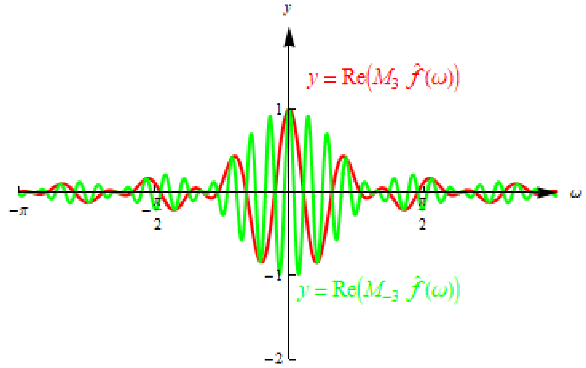


Figure 8. Real and imaginary part of the Fourier transform of the function $f(x)=\begin{cases}1, & x\in[1,2] \\ 0, & \text{otherwise}\end{cases}$.

For the Fourier transform of translation of $f(x)$ we have

$$\mathscr{F}\,(T_b f)(\omega) = M_{-b}\hat{f}(\omega) = i\,\frac{e^{-2\pi i\omega(b+2)} - e^{-2\pi i\omega(b+1)}}{2\pi\omega}$$
$$= \frac{\cos(\pi\omega(2b+3))\sin(\pi\omega)}{\pi\omega}$$
$$-i\,\frac{\sin(\pi\omega(2b+3))\sin(\pi\omega)}{\pi\omega}.$$

On Fig. 9 one can find the real and imaginary part of the Fourier transform of translation for $b = 3$ and $b = -3$, such that we use green color for $b = 3$ and red color $b = -3$.



a)

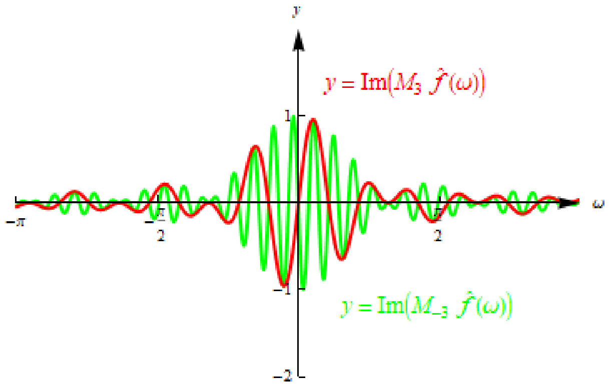


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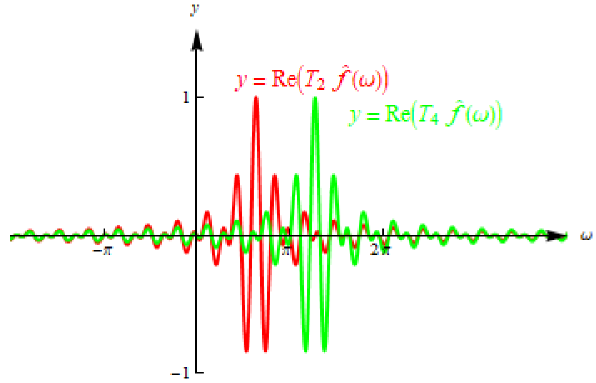
Figure 9. a) Real; b) Imaginary part of Fourier transform of translation of function $f(x) = \begin{cases} 1, & x \in [1,2] \\ 0, & \text{otherwise} \end{cases}$.

And, for the Fourier transform of $M_a f(x)$ we obtain

$$\mathscr{F}\,(M_a f)\,(\omega) = T_a\hat{f}(\omega)$$
$$= \frac{\cos(3\pi(\omega - a))\sin(\pi(\omega - a))}{\pi(\omega - a)}$$
$$-i\,\frac{\sin(3\pi(\omega - a))\sin(\pi(\omega - a))}{\pi(\omega - a)}.$$

The real and imaginary part of the Fourier transform of the modulation for $a = 2$ and $a = 4$, is given on Fig. 10, such that we use red color for $a = 2$ and green color for $a = 4$. Let we note that, when the parameter a increase, the graphic of real (resp. imaginary) part of Fourier transform of modulation translates on right, and when a decrease, the graphic of real (resp. imaginary) part of Fourier transform of modulation translates on left (see Fig. 10, a) (resp. Fig. 10, b)) .



a)

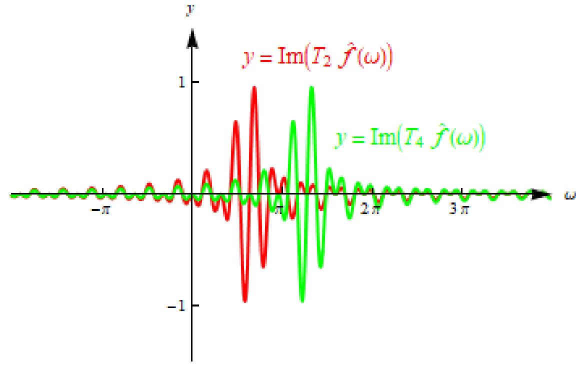


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Figure 10. a) Real; b) Imaginary part of Fourier transform of modulation of function $f(x) = \begin{cases} 1, & x \in [1,2] \\ 0, & \text{otherwise} \end{cases}$.

Example 7. The Fourier transform of the functions: $f(x) = 1$ and $f(x) = x$, $-\infty < x < \infty$; $f(x) = x^n$, $n > 1$, $-\infty < x < \infty$; $f(x) = e^{ax}$ and $f(x) = e^{-ax}$, $a > 0$, $-\infty < x < \infty$, does not exist because $\int_{-\infty}^{\infty} |f(x)|\,dx$ is not finite.

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